On (p,q)-analogue of Bernstein Operators (Revised)

M. Mursaleen, **Khursheed J. Ansari** and **Asif Khan**
Department of Mathematics, Aligarh Muslim University, Aligarh–202002, India
mursaleenm@gmail.com; ansari.jkhursheed@gmail.com; asifjnu07@gmail.com

Abstract

In the present article, we have given a corrigendum to our paper "On (p,q)-analogue of Bernstein operators" published in Applied Mathematics and Computation 266 (2015) 874-882.

Keywords and phrases: (p,q)- Bernstein operator; modulus of continuity; positive linear operator; Korovkin type approximation theorem.

AMS Subject Classifications (2010): 41A10, 41A25, 41A36, 40A30

1 Construction of Revised Operators

Mursaleen et. al [1] introduced (p,q)-analogue of Bernstein operators as

$$B_{n,p,q}(f;x)=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}x^k\prod_{s=0}^{n-k-1}(p^s-q^sx)\ \ f\left(\frac{[k]_{p,q}}{[n]_{p,q}}\right),\quad x\in[0,1]. \tag{1}$$

But $B_{n,p,q}(1;x)\neq 1$ for all $x\in[0,1]$. Hence, we are re-introducing our operators as follows:

$$B_{n,p,q}(f;x)=\frac{1}{p^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}p^{\frac{k(k-1)}{2}}x^k\prod_{s=0}^{n-k-1}(p^s-q^sx)\ \ f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right),\quad x\in[0,1]. \tag{2}$$

Note that for $p=1$, (p,q)-Bernstein operators given by (2) turn out to be q-Bernstein operators.

We have the following basic result:

Lemma 1. For $x\in[0,1],\ 0<q<p\leq 1$, we have

(i) $B_{n,p,q}(1;x)=\ 1$;
(ii) $B_{n,p,q}(t;x)=\ x$;
(iii) $B_{n,p,q}(t^2;x)=\ \frac{p^{n-1}}{[n]_{p,q}}x+\frac{q[n-1]_{p,q}}{[n]_{p,q}}x^2$.

Proof. (i)

$$B_{n,p,q}(1;x)=\frac{1}{p^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}p^{\frac{k(k-1)}{2}}x^k\prod_{s=0}^{n-k-1}(p^s-q^sx)=1.$$

(ii)

$$\begin{aligned}
B_{n,p,q}(t;x) &= \frac{1}{p^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{k(k-1)}{2}}x^k\prod_{s=0}^{n-k-1}(p^s-q^sx)\ \ \frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\\
&= \frac{1}{p^{\frac{n(n-3)}{2}}}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q} p^{\frac{(k+1)(k-2)}{2}}x^{k+1}\prod_{s=0}^{n-k-2}(p^s-q^sx)\\
&= \frac{x}{p^{\frac{(n-1)(n-2)}{2}}}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q} p^{\frac{k(k-1)}{2}}x^{k}\prod_{s=0}^{n-k-2}(p^s-q^sx) = x.
\end{aligned}$$

(iii)

$$\begin{aligned}
B_{n,p,q}(t^2;x) &= \frac{1}{p^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{k(k-1)}{2}}x^k\prod_{s=0}^{n-k-1}(p^s-q^sx)\ \ \frac{[k]^2_{p,q}}{p^{2k-2n}[n]^2_{p,q}}\\
&= \frac{1}{p^{\frac{n(n-5)}{2}}}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q} p^{\frac{(k+1)(k-4)}{2}}x^{k+1}\prod_{s=0}^{n-k-2}(p^s-q^sx)\ \frac{[k+1]_{p,q}}{[n]_{p,q}}\\
&= \frac{1}{p^{\frac{n(n-5)}{2}}[n]_{p,q}}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q} p^{\frac{(k+1)(k-4)}{2}}x^{k+1}\prod_{s=0}^{n-k-2}(p^s-q^sx)\ (p^k+q[k]_{p,q})\\
&= \frac{1}{p^{\frac{n(n-5)}{2}}[n]_{p,q}}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q} p^{\frac{k^2-k-4}{2}}x^{k+1}\prod_{s=0}^{n-k-2}(p^s-q^sx)\\
&\quad +\frac{q[n-1]_{p,q}}{p^{\frac{n(n-5)}{2}}[n]_{p,q}}\sum_{k=0}^{n-2}\left[\begin{array}{c} n-2 \\ k \end{array}\right]_{p,q} p^{\frac{(k+2)(k-3)}{2}}x^{k+2}\prod_{s=0}^{n-k-3}(p^s-q^sx)\\
&= \frac{p^{n-1}x}{[n]_{p,q}}\frac{1}{p^{\frac{(n-1)(n-2)}{2}}}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q} p^{\frac{k(k-1)}{2}}x^{k}\prod_{s=0}^{n-k-2}(p^s-q^sx)\\
&\quad +\frac{q[n-1]_{p,q}\ x^2}{[n]_{p,q}}\frac{1}{p^{\frac{(n-2)(n-3)}{2}}}\sum_{k=0}^{n-2}\left[\begin{array}{c} n-2 \\ k \end{array}\right]_{p,q} p^{\frac{k(k-1)}{2}}x^{k}\prod_{s=0}^{n-k-3}(p^s-q^sx)\\
&= \frac{p^{n-1}}{[n]_{p,q}}x+\frac{q[n-1]_{p,q}}{[n]_{p,q}}x^2.
\end{aligned}$$

Now, we prove Korovkin's type approximation theorem.

Theorem 1. Let $0<q_n<p_n\leq 1$ such that $\lim\limits_{n\to\infty}p_n=1$ and $\lim\limits_{n\to\infty}q_n=1$. Then for each $f\in C[0,1]$, $B_{n,p_n,q_n}(f;x)$ converges uniformly to f on $[0,1]$.

Proof. By the Korovkin's Theorem it suffices to show that

$$\lim_{n\to\infty}\|B_{n,p_n,q_n}(t^m;x)-x^m\|_{C[0,1]}=0,\quad m=0,1,2.$$

By Lemma 1(i)-(ii), it is clear that

$$\lim_{n\to\infty} \|B_{n,p_n,q_n}(1;x) - 1\|_{C[0,1]} = 0;$$

$$\lim_{n\to\infty} \|B_{n,p_n,q_n}(t;x) - x\|_{C[0,1]} = 0.$$

Using $q_n[n-1]_{p_n,q_n} = [n]_{p_n,q_n} - p_n^{n-1}$ and by Lemma 1 (iii), we have

$$\begin{aligned} |B_{n,p_n,q_n}(t^2;x) - x^2|_{C[0,1]} &= \left| \frac{p_n^{n-1}x}{[n]_{p_n,q_n}} + \left(\frac{q_n[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}} - 1 \right) x^2 \right| \\ &\leq \frac{p_n^{n-1}}{[n]_{p_n,q_n}} x + \frac{p_n^{n-1}}{[n]_{p_n,q_n}} x^2. \end{aligned}$$

Taking maximum of both sides of the above inequality, we get

$$\|B_{n,p_n,q_n}(t^2;x) - x^2\|_{C[0,1]} \leq \frac{2p_n^{n-1}}{[n]_{p_n,q_n}}$$

which yields

$$\lim_{n\to\infty} \|B_{n,p_n,q_n}(t^2;x) - x^2\|_{C[0,1]} = 0.$$

Thus the proof is completed.

2 Example

With the help of Matlab, we show comparisons and some illustrative graphics for the convergence of operators (2) to the function $f(x) = (x - \frac{1}{3})(x - \frac{1}{2})(x - \frac{3}{4})$ under different parameters.
From figure (1) we can observe that as the value the q increases, (p, q)-Bernstein operators given by (2) converges towards the function.

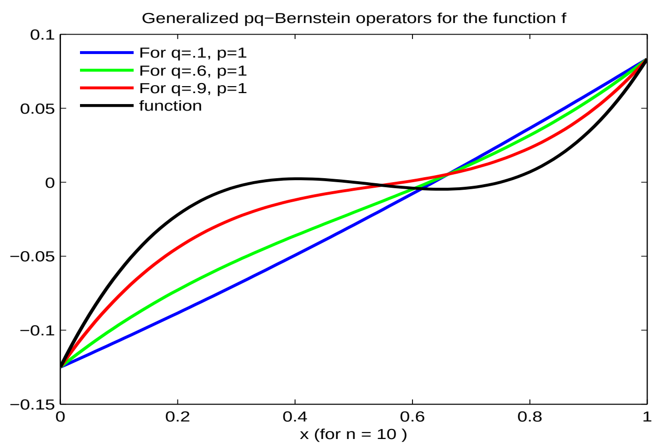

Figure 1:

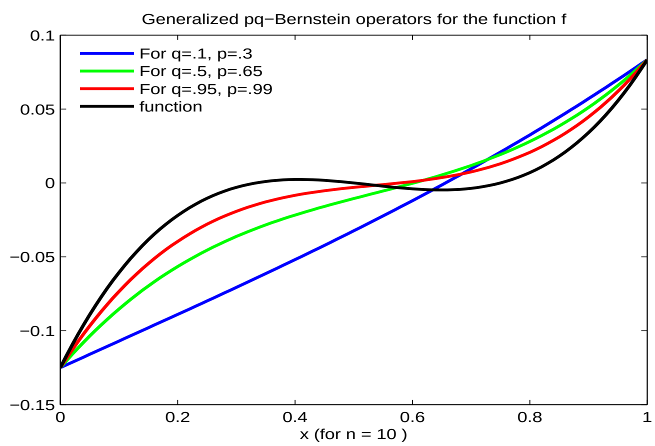

Figure 2:

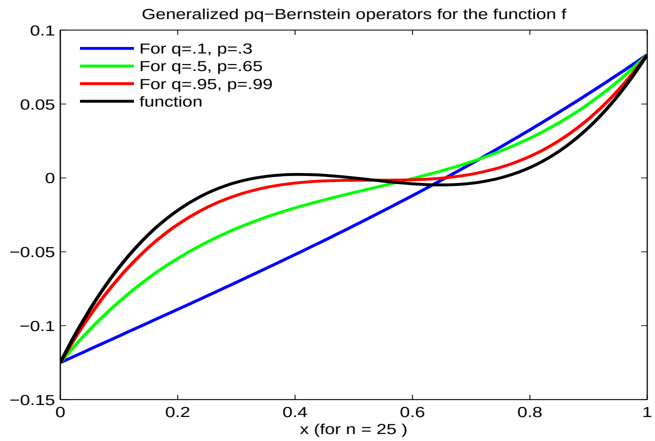

Figure 3:

In comparison to figure 2, as the value the n increases, operators given by (2) converge towards the function which is shown in figure 3. Also, from figure 2, it can be observed that as the value of p, q approaches towards 1 provided $0 < q < p \leq 1$, operators converge towards the function.

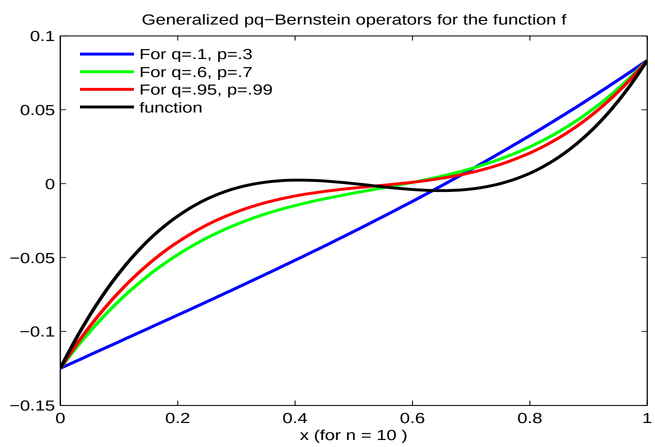

Figure 4:

Similarly for different values of parameters p, q and n convergence of operators to the function is shown in figure 4 and figure 5.

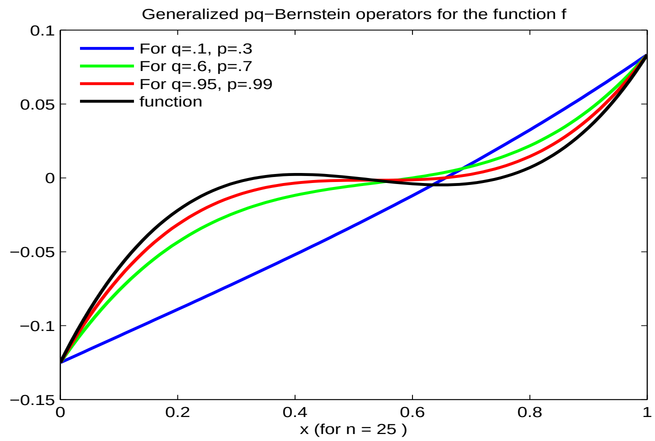

Figure 5:

References

[1] M. Mursaleen, Khursheed J. Ansari, Asif Khan, On (p, q)-analogue of Bernstein operators, Applied Mathematics and Computation, 266 (2015) 874-882.